A Three-Dimensional Anisotropic Viscoelastic Generalized Maxwell Model for Ageing Wood Composites

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Wood subject to varying thermal and hygrometric condition is fairly common, understanding its behaviour is important for serviceability, durability and design of structures.

The starting point for us: the dimensional stability of a kitchen cabinet door :

- effect of the varying hygrometric conditions,
- sensitivities to deviation of orientation of the grain,
- etc,

Wood/wood composite are porous, anisotropic.

Ageing = time dependency of the properties of the material

The mechanical properties of wood depends on moisture content and temperature: hygrothermal ageing.

Viscoelasticity = between fluid (viscous) and solid (elastic)

Wood is an anisotropic ageing viscoelastic material.

No elaborate physics (i.e. coupled equations) since it leads to adding terms to the constitutive law (ex. thermal expansion, shrinking/swelling,...).

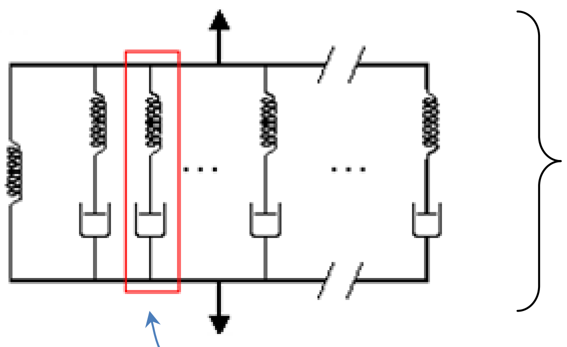
Ageing is represented by a known scalar function M

The function M is independent of the temperature.

M is an external variable (thermodynamics)

Rheological viscoelastic model

The generalized Maxwell model is a viscoelastic model satisfying the thermodynamic principles. It is usually defined using a set of springs and dashpots (dampers).



The generalized Maxwell model is composed of a spring and a group of Maxwell cell in parallel.

We want a “generalized Maxwell model” (GMM) which is

- three-dimensional, anisotropic and thermodynamically admissible.

- based on the Prony series, it should take into account ageing (isothermal or not depending on T).

- easy to integrated into pre-existing finite element (FE) code and avoid imposing limitation on the numerical method.

1. Prony coefficients for the ageing 3D GMM.
   - Ageing GMM in the uniaxial case
   - Connection with the Prony series in 3D.
2. Simplifying the use of the Prony series
   - A parameterization of ageing in a Prony series.
   - Differential formulation for a “good” form of the constitutive law

Based on:

- Zocher (1997) : incremental ageing 3D model
- Dubois (2005) : ageing and thermodynamics
- Poon (1998) : differential formulation

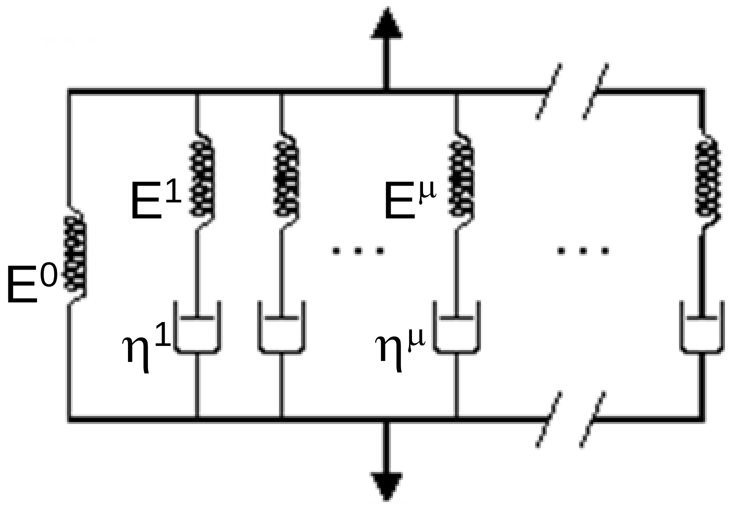
- State variables: displacement vector u, stress tensor σ .
- f_b the body force and the strain tensor

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad 1 \leq i, j \leq 3$$

- The governing equation for a linear quasi-static (the inertial term is neglected) viscoelastic body

$$-\nabla \cdot \sigma = f_b$$

- initial and boundary conditions on u and σ.



A uniaxial ageing GMM composed of N Maxwell cells. For each cells we have an ageing spring and damper (dashpot).

$E^{\mu}(t)=E^{\mu}(M(t))$, $\eta^{\mu}(t)=\eta^{\mu}(M(t))$: stiffness and viscosity $\mu=1,\ldots,N$.

Constitutive law in that case ?

What is the impact of ageing on the constitutive law of the spring and damper ?

Following Bazant (1979) and Gril (1988) the constitutive law of an ageing spring of constant E(t) is

$$\dot{\sigma} = \begin{cases} E\dot{\varepsilon} & \text{if } E' \geq 0 \text{ (hardening)} \\ E\dot{\varepsilon} + E'\varepsilon & \text{if } E' \leq 0 \text{ (softening)} \end{cases}$$

$$\dot{\sigma} = E\dot{\varepsilon} + \min(0, E')\varepsilon = E\dot{\varepsilon} + (E')^{-}\varepsilon$$

Mecanosorption, "hygro-locking": at constant strain, increasing the stifness as no effect on the stress

For an ageing dashpot, Newton's law is sufficient provided that the viscosity $\eta(t)$ is positive at all time:

$$\sigma = \eta(t)\dot{\varepsilon} \quad \eta(t) \geq 0$$

From the spring and damper the (thermodynamically valid) constitutive law for a uniaxial ageing GMM with N cells is

$$\sigma = \sigma^0 + \sum_{\mu=1}^{N} \int_0^t E^\mu(s) e^{-\int_s^t \lambda^\mu(r)dr} \frac{d\varepsilon}{dt} ds \qquad \lambda^\mu(t) = \frac{E^\mu(t)}{\eta^\mu(t)} - \frac{(E^{\mu\prime}(t))^-}{E^\mu(t)}$$

where σ^0 is the elastic stress, E^μ and η^μ are the stiffness and viscosity of the μ^{th} cell.

Considering the stresses in direction i j for every uniaxial strain in direction k l and using the superposition principle:

$$\sigma_{ij}(t) = \sigma_{ij}^{0} + \sum_{\mu=1}^{N} \sum_{k\,l} \int_{0}^{t} E_{ijkl}^{\mu}(t) e^{-\int_{s}^{t} \lambda_{ijkl}^{\mu}(t)dr} \frac{d\varepsilon_{kl}}{dt} ds$$

$$\lambda_{ijkl}^{\mu}(t) = \frac{E_{ijkl}^{\mu}(t)}{\eta_{ijkl}^{\mu}(t)} - \frac{(E_{ijkl}^{\mu}{}'(t))^{-}}{E_{ijkl}^{\mu}(t)}$$

$E_{ijkl}^{\mu}(t), \eta_{ijkl}^{\mu}(t)$ the stiffness and viscosity tensor for $\mu=1,\ldots,N$

σ is related to the history of the strain ε (hereditary relation).

$$\sigma_{ij}(t) = \sigma_{ij}^{0} + \sum_{\mu=1}^{N} \sum_{k\,l} \int_{0}^{t} E_{ijkl}^{\mu}(t) e^{-\int_{s}^{t} \lambda_{ijkl}^{\mu}(t)dr} \frac{d\varepsilon_{kl}}{dt} ds$$

How do we use this relation?

The best scenario: transform the constitutive law in a linear relation between σ and ε at each time step.

Given a reference state of ageing M_{ref} and the stiffness and viscosity tensor at M_{ref} being $E_{ijkl}^{\mu,ref}, \eta_{ijkl}^{\mu,ref}$

We introduce the parametrized ageing:

$$b_{ijkl}^{\mu}(M(t)) = b_{ijkl}^{\mu}(t),\ b_{ijkl}^{\mu}(M_{ref}) = 1 \qquad E_{ijkl}^{\mu}(t) = b_{ijkl}^{\mu}(t) E_{ijkl}^{\mu,ref}$$

$$l_{ijkl}^{\mu}(M(t)) = l_{ijkl}^{\mu}(t) \geq 0,\ l_{ijkl}^{\mu}(M_{ref}) = 1 \qquad \eta_{ijkl}^{\mu}(t) = l_{ijkl}^{\mu}(t) \eta_{ijkl}^{\mu,ref}$$

3D spring:
- Isotropic: max. of 2 distinct parameters
- Isotropic plane: max. of 5 distinct parameters
- Orthotropic: max. of 9 distinct parameters

$$\zeta^{\mu}_{ijkl}(t) = \int_0^t \frac{b^{\mu}_{ijkl}(r)}{l^{\mu}_{ijkl}(r)}dr, \quad \lambda^{\mu,ref}_{ijkl} = \frac{E^{\mu,ref}_{ijkl}}{\eta^{\mu,ref}_{ijkl}} \quad \lambda^{\mu}_{ijkl}(t) = \frac{b^{\mu}_{ijkl}(t)}{l^{\mu}_{ijkl}(t)}\lambda^{\mu,ref}_{ijkl} - \frac{(b^{\mu}_{ijkl}{}'(t))^-}{b^{\mu}_{ijkl}(t)}$$

$$\Delta\zeta^{\mu}_{ijkl}(t,s) = \int_s^t \frac{b^{\mu}_{ijkl}(r)}{l^{\mu}_{ijkl}(r)}dr, \qquad \Delta\psi^{\mu}_{ijkl}(t,s) = e^{\int_s^t \frac{(b^{\mu}_{ijkl}{}'(r))^-}{b^{\mu}_{ijkl}(r)}dr} \leq 1$$

$$s^{\mu}_{ijkl}(t) = b^{\mu}_{ijkl}(t)(\varepsilon_{kl}(t) - \varepsilon_{kl}(0)) - \int_0^t \Delta\psi^{\mu}_{ijkl}(t,s)b^{\mu}_{ijkl}(s)e^{-\lambda^{\mu,ref}_{ijkl}\Delta\zeta^{\mu}_{ijkl}(t,s)}\frac{d\varepsilon_{kl}}{dt}ds$$

ζ^{μ}_{ijkl} is the reduced time, $\lambda^{\mu,ref}_{ijkl}$ the reference relaxation time,

$\Delta\psi^{\mu}_{ijkl}$ a softening factor (=1 if hardening/no ageing on [s,t]).

$s^{\mu}_{ijkl}(t)$ a pseudo-strain “simplifying” the constitutive law

$$\sigma_{ij}(t) = \sigma_{ij}^{0} + \sum_{\mu=1}^{N} \sum_{kl} E_{ijkl}^{\mu}(t)(\varepsilon_{kl}(t) - \varepsilon_{kl}(0)) - \sum_{\mu=1}^{N} \sum_{kl} E_{ijkl}^{\mu,ref} s_{ijkl}^{\mu}(t)$$

Two approaches to implement numerically

- Incremental approach with a linear approx. of the strain over the time step. Need small time step for accuracy and numerical stability (Bazant (1979), Zocher (1997), Dupuis (2005)).
- Incremental approach with a differential equation for the discretization of s_{ijkl}^{μ} (Poon (1999))

We favour the ODE approach, we derive the pseudo-strain, and get a system as a constitutive law

$$\sigma_{ij}(t) = \sigma_{ij}^{0} + \sum_{\mu=1}^{N} \sum_{kl} E_{ijkl}^{\mu}(t)(\varepsilon_{kl}(t) - \varepsilon_{kl}(0)) - \sum_{\mu=1}^{N} \sum_{kl} E_{ijkl}^{\mu,ref} s_{ijkl}^{\mu}(t)$$

$$\begin{cases} \dot{s}_{ijkl}^{\mu}(t) = \left(\dot{b}_{ijkl}^{\mu}(t) + b_{ijkl}^{\mu}(t)\lambda_{ijkl}^{\mu}(t)\right)(\varepsilon_{kl}(t) - \varepsilon_{kl}(0)) - \lambda_{ijkl}^{\mu}(t) s_{ijkl}^{\mu}(t) \\ s_{ijkl}^{\mu}(0) = 0 \end{cases}$$

An arbitrary time integration scheme can be used for s_{ijkl}^{μ} based on accuracy and stability.

For the one step θ-schemes (denoting $f(t_{n+1})=f^{n+1}$)

$$E^{1,n+1}_{ijkl} = \sum_{\mu=1}^{N} E^{\mu,ref}_{ijkl} \left(b^{\mu,n+1}_{ijkl} - \Delta t_{n+1}\theta \frac{\dot{b}^{\mu,n+1}_{ijkl} + b^{\mu,n+1}_{ijkl}\lambda^{\mu,n+1}_{ijkl}}{1+\Delta t_{n+1}\theta\lambda^{\mu,n+1}_{ijkl}} \right)$$

$$E^{2,n+1}_{ijkl} = \sum_{\mu=1}^{N} E^{\mu,ref}_{ijkl} \Delta t_{n+1}(1-\theta) \frac{\dot{b}^{\mu,n}_{ijkl} + b^{\mu,n}_{ijkl}\lambda^{\mu,n}_{ijkl}}{1+\Delta t_{n+1}\theta\lambda^{\mu,n+1}_{ijkl}}$$

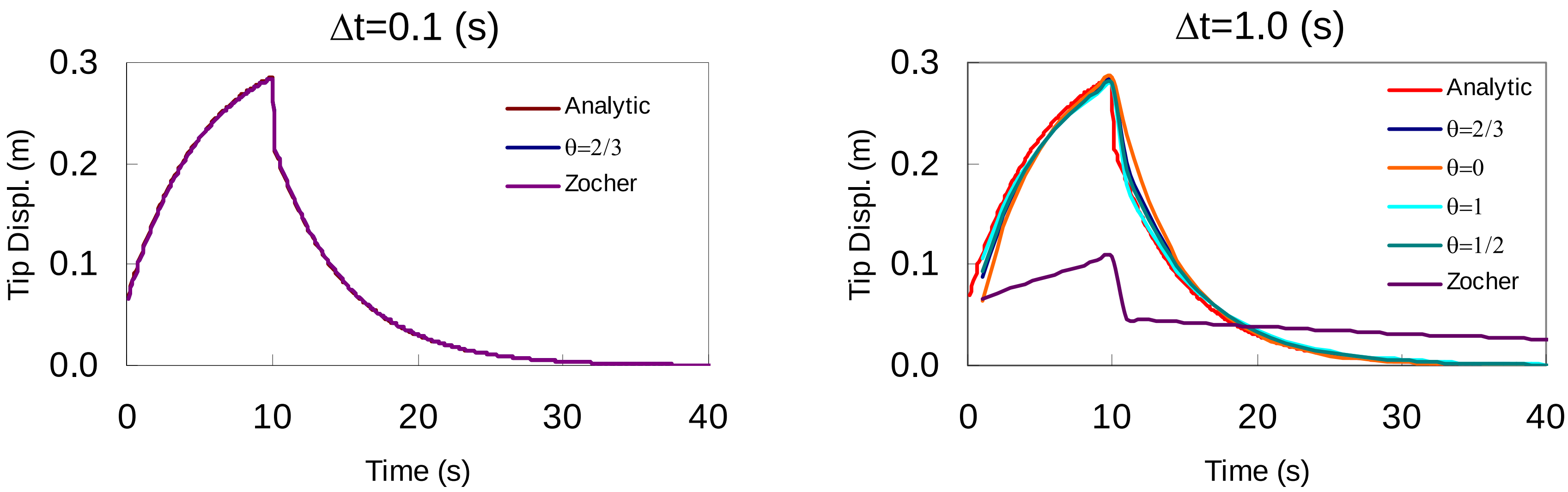
$$s^{n+1}_{ij} = \sum_{k\,l}\sum_{\mu=1}^{N} (E^{\mu,ref}_{ijkl} b^{\mu,n+1}_{ijkl} - E^{1,n+1}_{ijkl}) s^{\mu,n}_{ijkl}$$

θ	
0	Explicit Euler
1	Implicit Euler
1/2	Crank-Nicholson
2/3	Galerkin

We have the discrete constitutive law

$$\sigma^{n+1}_{ij} = E^{1,n+1}_{ijkl}(\varepsilon^{n+1}_{kl} - \varepsilon_{kl}(0)) + \sigma^{0}_{ij} - s^{n+1}_{ij} - E^{2,n+1}_{ijkl}(\varepsilon^{n}_{kl} - \varepsilon_{kl}(0))$$

Isotropic non-ageing cantilever beam subjected to a tip load (20 m long, 1 m^2 cross-section, see Zocher 1997 for details).



The impact of the choice of time step can be quite important even in the isotropic case.

We have a numerical algorithm for a three-dimensional anisotropic ageing viscoelastic model. The model satisfy the thermodynamic and is easily integrated in pre-existing FE code.

The numerical results are in accordance with the literature. More importantly, we had a noticeable gain in performance (larger time step) for comparable results (Zocher 1997).

Our main difficulty is the determination of parameters for the three-dimensional case. We are presently doing experimental work on maple.